COMPUTATIONAL LEARNING IN DYNAMIC LOGICS

DAY 4: ITERATED BELIEF REVISION AND LEARNABILITY

Nina Gierasimczuk and Caleb Schultz Kisby

@NASSLLI, June 2025

Course Homepage:

`https://sites.google.com/view/nasslli25-learning-in-del`

PLAN FOR TODAY

1. Learnability in Epistemic Spaces

2. Learning Power of Belief Revision Operators

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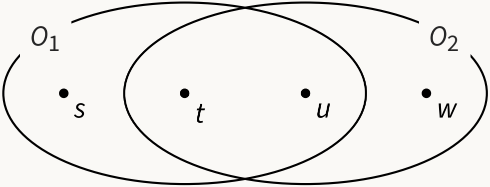
- a topological characterisation of learnability and solvability;
- a modal dynamic logic of learnability.

Definition

An **epistemic space** is a pair $\mathbb{S} = (S, O)$ consisting of a state space S and a set of observables $\mathcal{O} \subseteq \mathcal{P}(S)$, both at most countable.



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We assume that data streams are sound and complete.

Definition

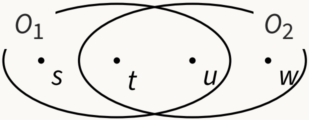
Let $\mathbb{S} = (S, O)$ be an epistemic space and let $O_0, \ldots, O_n \in \mathcal{O}$.

A **learner** is a function L that on the input of $\mathbb{S}$ and data sequence $(O_0, \ldots, O_n)$ outputs some set of worlds

$$L(\mathbb{S}, (O_0, \ldots, O_n)) \subseteq S$$

We call this the learner's **conjecture**.

AN INTUITION ABOUT SEPARABILITY BY OBSERVATIONS



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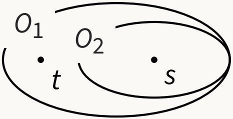
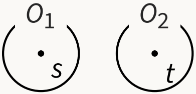


(b) weakly separated space $T0$



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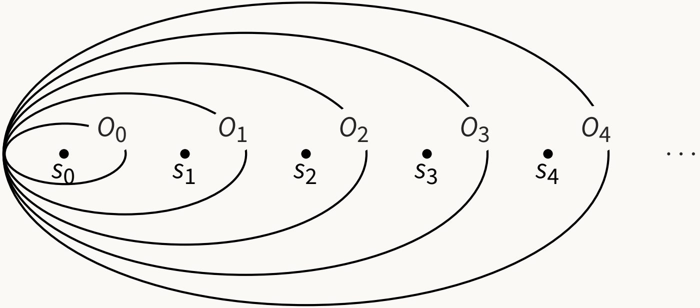
there is $n \in \mathbb{N}$ such that:

$$L(\mathbb{S}, \vec{O}[k]) = \{s\} \text{ for all } k \geq n.$$

An epistemic space $\mathbb{S}$ is **learnable** if it is learnable by a learner L.

EXAMPLE OF A LEARNABLE SPACE

Let $\mathbb{S} = (S, O)$ such that $S = \{s_n \mid n \in \mathbb{N}\}$, $\mathcal{O} = \{O_i \mid i \in N\}$, and for any $k \in \mathbb{N}$, $O_k = \{s_i \mid 0 \le i \le k\}$. $\mathbb{S}$ is learnable.



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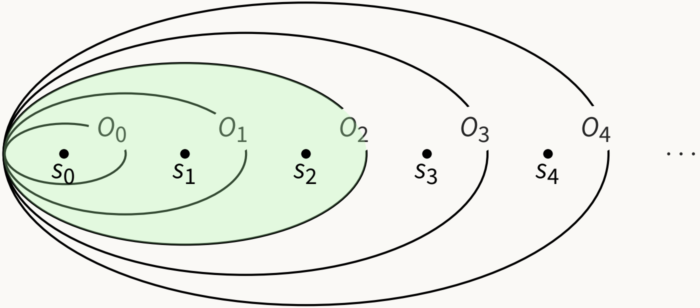

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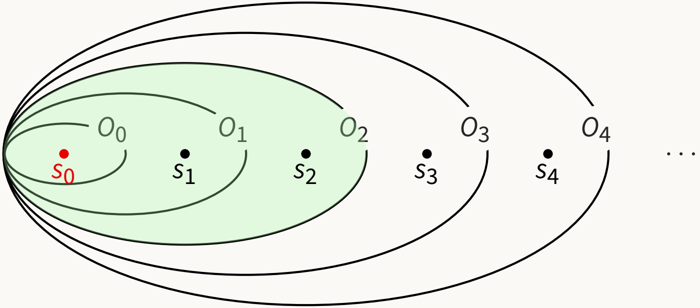

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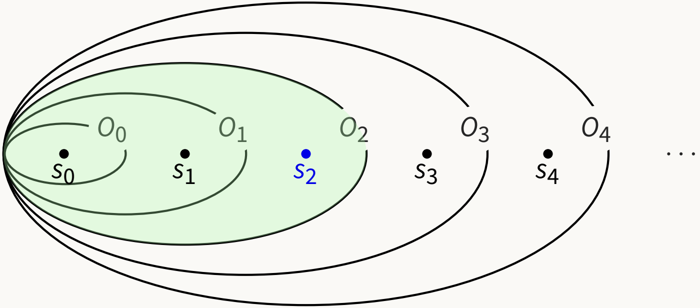

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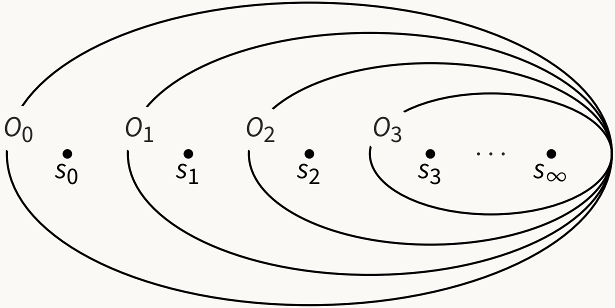

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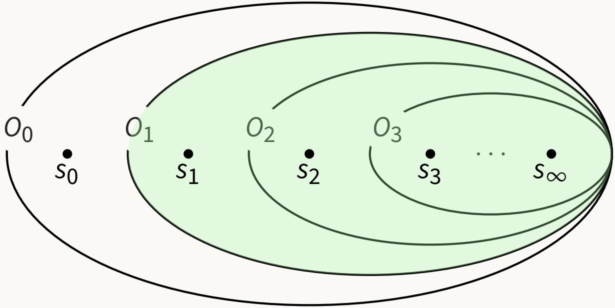

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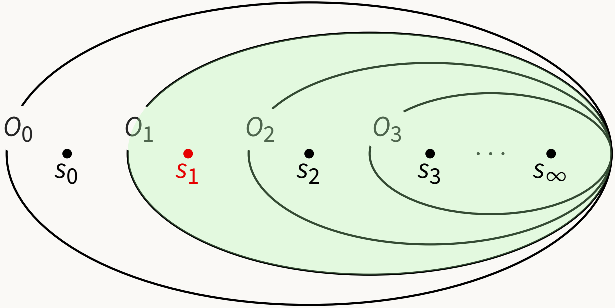

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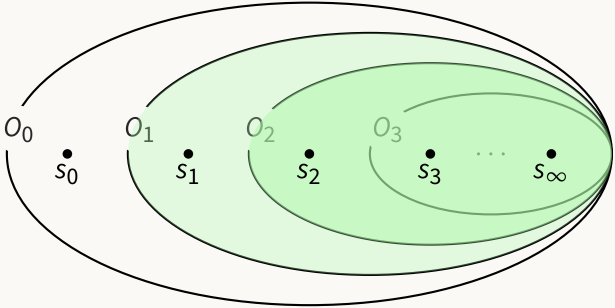

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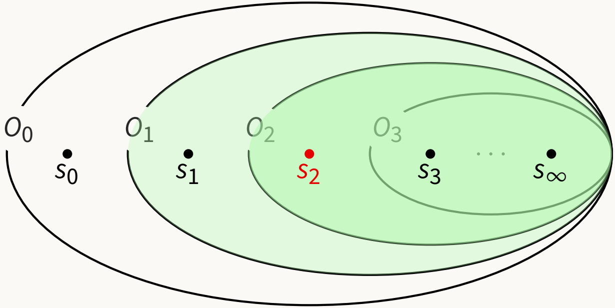

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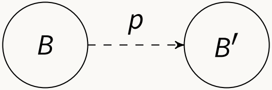

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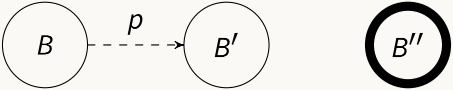

Truth-tracking!

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Knowledge and Belief

$$\mathbb{B}_{\mathbb{S}} \vDash Kp \quad \text{iff} \quad S \subseteq p$$
$$\mathbb{B}_{\mathbb{S}} \vDash Bp \quad \text{iff} \quad min_{\preceq} S \subseteq p.$$

Definition

A **belief-revision method** is a function R that, for any plausibility space $\mathbb{B}_{\mathbb{S}} = (S, O, \leq)$ and any observation O outputs a new plausibility space:

$$R(\mathbb{B}_{\mathbb{S}}, O) := (S', O, \leq').$$

BELIEF-REVISION METHODS

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A belief revision R can be iterated in the following way:

$$R(\mathbb{B}_{\mathbb{S}}, \sigma * O) := R(R(\mathbb{B}_{\mathbb{S}}, \sigma), O)$$

where σ is a finite sequence of observations.

ADDITIONAL COMMENTS ON BELIEF-REVISION METHODS

We will look at three belief-revision methods we are now familiar with

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- Philosophy of Science: Ockham's razor.

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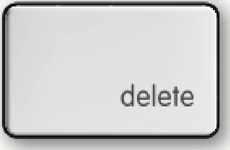

LEXICOGRAPHIC UPGRADE

- **Lexicographic upgrade** rearranges the preorder by putting all worlds satisfying the observation to be more plausible than others.

MINIMAL UPGRADE

- **Minimal upgrade** rearranges the preorder by making only the most plausible states satisfying the observation more plausible than all others, leaving the rest of the preorder the same.

LEARNING VIA BELIEF REVISION

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Every belief-revision method R, together with a prior plausibility $\leq$ generates in a canonical way a learning method $L_R^{\leq}$ called a **belief-revision-based learning method**, and given by:

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A. Baltag, N. Gierasimczuk, S. Smets. Truth tracking by belief revision. Studia Logica 2018.

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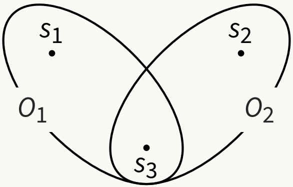

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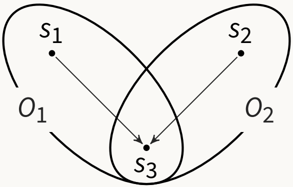

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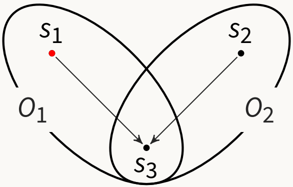

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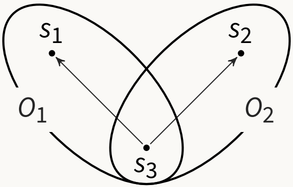

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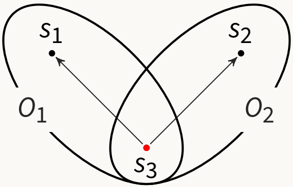

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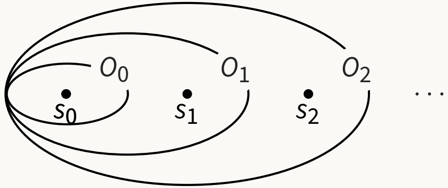

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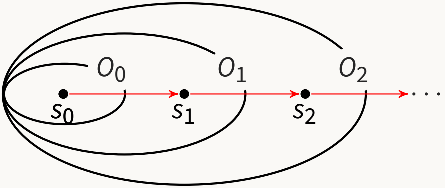

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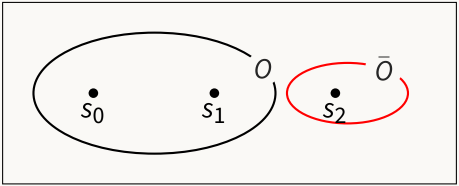
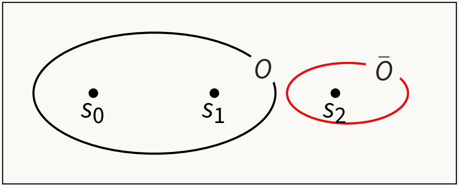


Definition

Let $\mathbb{S} = (S, \mathcal{O})$ be a negation-closed epistemic space. A stream $\vec{O}$ is **fair** with respect to the world s if $\vec{O}$ is complete wrt s, and contains only finitely many observations O, s.t. $s \notin O$ and every such error is eventually corrected in $\vec{O}$.

IS $\neg O$ OBSERVABLE?

An epistemic space $\mathbb{S} = (S, \mathcal{O})$ is **negation-closed** iff if $O \in \mathcal{O}$, then $\bar{O} \in \mathcal{O}$.



Definition

Let $\mathbb{S} = (S, \mathcal{O})$ be a negation-closed epistemic space. A stream $\vec{O}$ is **fair** with respect to the world s if $\vec{O}$ is complete wrt s, and contains only finitely many observations O, s.t. $s \notin O$ and every such error is eventually corrected in $\vec{O}$.

EXTENDED UNIVERSALITY RESULTS

	Conditioning	Lexicographic	Minimal
Positive	YES	YES	NO
Positive and Negative	YES	YES	NO
Fair Streams	NO	YES	NO

END OF LECTURE 3

Thank you!